Math 332H • May 7, 2009

Final Examination

This is a closed-book exam; neither notes nor calculators are allowed.

You can choose between problems 2 and 2', 3 and 3' (choose one in each pair)

1) (20pts) For each, find **all distinct** values of z, in Cartesian form, and simplify as much as possible:

(a) $z = \left|(-1)^i\right|$ (b) $e^{iz} = -4i$ (c) $z = (1-i)^{2/3}$

2) (12pts) Sketch the mapping of the region $0 \le \text{Arg}\, z \le \pi/4$, $1 \le |z| \le 2$, under the transformation $w = i/z$

2') (12pts) Show that the function $u(x, y) = 2x^2 - 3y - 2y^2 + 5$ is harmonic, find its harmonic conjugate, $v(x, y)$, and express the analytic function $f(x, y) = u(x, y) + i\, v(x, y)$ as a function of z. Assume that $f(i) = i$.

3) (12pts) Find the two different series representations for the function

$f(z) = \dfrac{1}{z(z+2i)^2}$ centered at $z = 0$ (use the full Σ notation), and show the two regions in the complex plane where each series is valid [hint: you may need term-by-term differentiation to derive the series for $1/(1+w)^2$]

3') (12pts) Find the singularity type, the residue and the first three non-zero terms in the series for the following function at the specified point:

$$f(z) = \frac{e^z - \cos z}{z \sin z} \quad \text{at} \quad z = 0$$

4) (28pts) Find the following contour integrals, using the method of residues where possible. The integration contour is a circle of radius R=5 around the origin, in the positive direction:

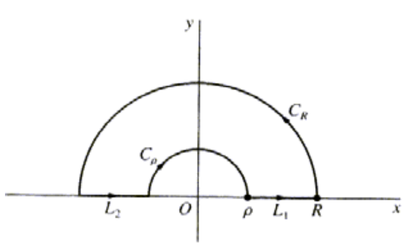
(a) $\displaystyle\oint_{C_{R=5}} \frac{dz}{e^z \sin z}$ (b) $\displaystyle\oint_{C_{R=5}} \frac{\sinh z\, dz}{(z-i-3)^2}$ (c) $\displaystyle\oint_{C_{R=5}} \frac{dz}{(e^z-1)^2}$ (d) $\displaystyle\oint_{C_{R=5}} \text{Log}\, z\, dz$

5) (28pts) Calculate **any two** of the following three integrals**, justifying each step.**

(a) $\displaystyle\int_0^\infty \frac{x^2 \cos(2x)}{x^4+4}\,dx$ (b) $\displaystyle\int_0^\infty \frac{\sin(2x)}{x}\,dx$ (c) $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{x^2+2x+1}\,dx$

In (b) and (c), use the contours shown below

(b)



(c)